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Aspects of the hypermultiplet moduli space in string duality

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Abstract: A type IIA string (or F-theory) compactified on a Calabi–Yau threefold is believed to be dual to a heterotic string on a K3 surface times a 2-torus (or on a K3 surface). We consider how the resulting moduli space of hypermultiplets is identified between these two pictures in the case of the $E_8 \times E_8$ heterotic string. As examples we discuss SU(2)-bundles and G_2-bundles on the K3 surface and the case of point-like instantons. We are lead to a rather beautiful identification between the integral cohomology of the Calabi–Yau threefold and some integral structures on the heterotic side somewhat reminiscent of mirror symmetry. We discuss the consequences for probing nonperturbative effects in the both the type IIA string and the heterotic string.

Keywords: String Duality, Differential and Algebraic Geometry.

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Contents

1. Introduction

It has long been supposed that a type IIA string suitably compactified on a Calabi–Yau threefold is dual to a heterotic string suitably compactified on a product of a K3 surface and a 2-torus [1]. The resulting physics in four dimensions consists of a theory with $N = 2$ supersymmetry. At least locally, the moduli space for these theories is a product of a special Kähler manifold corresponding to moduli in the vector supermultiplets, and a quaternionic Kähler manifold corresponding to moduli in the hypermultiplets. The vector moduli space has been fairly well understood for some time now (see, for example, [2] and references therein). The hypermultiplet space has been somewhat more awkward to understand (although some progress has been made, see for example [3]).

An almost equivalent problem arises in F-theory. If one compactifies F-theory on a Calabi–Yau threefold it can be dual to the heterotic string compactified on a K3 surface. Our resulting theory is now an N=1 theory in six dimensions. There is still a quaternionic Kähler moduli space corresponding to the moduli in the hypermultiplets. Indeed, if this six-dimensional theory is compactified on a 2-torus, we recover the $N = 2$ theory in four dimensions above with an unchanged hypermultiplet moduli space.[1]

[1]Except that a few parameters, such as the volume of the K3 surface, are reinterpreted in the F-theory picture. We use the type IIA language throughout this paper.

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Let us consider the type IIA string compactified on a particular Calabi–Yau threefold, X. The hypermultiplet moduli space is then composed of deformations of the complex structure of X, together with Ramond-Ramond moduli living in the "intermediate Jacobian" of X, and the dilaton and axion. On the other hand, in the heterotic string picture the hypermultiplet moduli space consists of deformations of a particular bundle on the K3 surface together with deformations of the underlying K3 surface itself. The central question in understanding the moduli space of hypermultiplets is to know exactly how to match this data between the type IIA and heterotic picture.

Thus somehow given a Calabi–Yau threefold together with its intermediate Jacobian, we should be able to "derive" some K3 surface with a particular bundle. This is not a property of known classical geometry. A very similar statement could be made concerning mirror symmetry. In this case a type IIA string compactified on a Calabi–Yau threefold X is dual to a type IIB string compactified on the mirror Calabi–Yau threefold Y. Again one did not know classically how to "derive" Y given X.

The key idea in mirror symmetry was to think in terms of large radius limits and "large complex structure" limits. Similarly this paper will dwell on the analogous boundary in the hypermultiplet moduli space. This latter question is much more interesting than that of mirror symmetry. Indeed this whole subject of exploring the moduli space of hypermultiplets looks like a much richer version of the story of mirror symmetry.

The main purpose of the paper is to identify an integer structure on the type IIA side, coming from $H^3(X,\mathbb{Z})$, with an integer structure on the heterotic side coming from $H^2(\mathrm{K3},\mathbb{Z})$ and the bundle data. This is a very useful handle for picturing how the moduli space of hypermultiplets is mapped between the type IIA string and the heterotic string.

In section 2 we will discuss the boundary of the moduli space we need to study to remove all quantum effects. This includes a discussion of the B-field on the heterotic K3 surface. In section 3 we will discuss the moduli space of bundles on a K3 surface in the language of string duality. Much of this section follows from the work of Friedman, Morgan and Witten [4] but we explain the construction in detail for a few examples. Finally in section 4 we discuss the integral structure and some of its consequences.

2. The stable degeneration

2.1. The geometry of the degeneration

The type IIA string compactified on a Calabi–Yau threefold, X, has the following moduli:

1. The dilaton, which governs the string coupling, and the axion coming from dualizing a two-form in four dimensions. Together these form a complex field, Φ_{IIA}. We assert that $\Phi_{\mathrm{IIA}} \to -\infty$ is the weakly-coupled limit of this type IIA string theory.

2. A metric which is determined by the complex structure of X and the cohomology class of its Kähler form. When all length scales in X are large with respect to the string tension, this metric is Ricci-flat.

3. A B-field which takes values in $H^2(X, \mathbb{R}/\mathbb{Z})$ (at least when X is large).

4. Ramond-Ramond fields, $R \in H^{\mathrm{odd}}(X, \mathbb{R}/\mathbb{Z})$.[2]

The Kähler form modulus and the B-field pair up to form the "complexified Kähler form". The moduli space of vector moduli, $\mathscr{M}_V$, is parametrized by this complexified Kähler form. There are world-sheet instanton corrections to this moduli space however and the complexified Kähler form is only a valid coordinate near the large radius limit of X. We will also refer to these world-sheet instantons as α' corrections as their scale is determined by the string tension. $\mathscr{M}_V$ is a special Kähler manifold (at least where it is smooth away from any boundary points).

The other fields, namely Φ_{IIA}, R, and the complex structure of X parameterize the hypermultiplet moduli space. Again this moduli space is subject to corrections except that this time the corrections are due to string coupling. The above parameters are only good coordinates when we are near the limit of a weakly-coupled string, $\Phi_{\mathrm{IIA}} \to -\infty$. The moduli space built from these parameters is a quaternionic Kähler manifold and we denote it $\mathscr{M}_H$.

Now let us consider the heterotic string. There are two types of heterotic strings which we label by $\mathcal{G}_0$. $\mathcal{G}_0$ is either equal to $\mathrm{Spin}(32)/\mathbb{Z}_2$ or $(E_8 \times E_8) \rtimes \mathbb{Z}_2$, where the latter case is usually referred to as the $E_8 \times E_8$ heterotic string.[3]

The heterotic string compactified on a product of a K3 surface, S_H, and a 2-torus, E_H, has the following moduli:

1. The dilaton, which governs the string coupling, and the axion coming from dualizing a two-form in four dimensions. Together these form a complex field, Φ_{Het}. We assert that $\Phi_{\mathrm{Het}} \to -\infty$ is the weakly-coupled limit of this heterotic string theory.

2. A Ricci-flat metric on $S_H \times E_H$.

3. A B-field which takes values in $H^2(S_H \times E_H, \mathbb{R}/\mathbb{Z})$ (at least when S_H and E_H are large).

4. A $\mathcal{G}_0$-bundle on $S_H \times E_H$ with a connection satisfying the Yang-Mills equation.

[2]It has been suggested [5] that the charges of BPS states lie in $H^*(X, \mathbb{Z}/N)$ for some integer $N > 1$. This may cause the RR moduli to live in $H^{\mathrm{odd}}(X, \mathbb{R}/(\mathbb{Z}/N))$. If this is the case then some of the statements in this paper need to be modified to take this isogeny into account.

[3]The extra $\mathbb{Z}_2$ arises as the possibility of exchanging the two E_8's. It is essential to include this possibility in order to understand duality between the two heterotic strings correctly [6, 7].

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In order to proceed further we need to make a restriction on the type of this bundle. Let $\mathcal{G}_0 \supset \mathcal{G}_S \times \mathcal{G}_E$ and assume that the $\mathcal{G}_0$-bundle on $S_H \times E_H$ may be viewed as a product of a $\mathcal{G}_S$-bundle on S_H and a $\mathcal{G}_E$-bundle on E_H.

The vector multiplet moduli space, $\mathscr{M}_V$, can now be viewed as being parametrized by Φ_{Het}, the moduli of the $\mathcal{G}_E$-bundle on E_H and by the deformations of the metric and B-field on E_H. This moduli space is subject to corrections from the string coupling and identifications from T-dualities for E_H. The above parameters are only seen as good coordinates when $\Phi_{\text{Het}} \to -\infty$ and the area of E_H is large.

The hypermultiplet moduli space, $\mathscr{M}_H$, can be viewed as being parametrized by the moduli of the $\mathcal{G}_S$-bundle on S_H and by the deformations of the metric and B-field on S_H. The corrections to this moduli space are, as yet, not fully understood but we will see that this picture is probably prone to α' corrections. What is clear is that the above parameters are good coordinates when the volume of S_H is large.

The main purpose of this paper will be to try to match the type IIA (or F-theory) picture of the moduli space $\mathscr{M}_H$ with that of the heterotic string. Going to the limits in which we remove the quantum corrections, we may match our coordinate systems and obtain a map between a Calabi–Yau threefold, X and a bundle on a K3 surface, S_H. In order to do this we need to take $\Phi_{\text{IIA}} \to -\infty$ and S_H to have a large volume.

The first thing we should worry about are geometrical restrictions on X for the above duality to be possible. We want to impose the conditions that

1. We do have a heterotic string dual to the type IIA string on X which can be recognized as such in a simple way.

2. The $\mathcal{G}_0$-bundle does indeed factorize nicely as a $\mathcal{G}_S$-bundle and a $\mathcal{G}_E$-bundle as desired.

It can be shown (see, for example, [2]) that these conditions amount to asking that X has a dual fibration — firstly a K3-fibration $p: X \to B$ and as an elliptic fibration $\pi_F: X \to \Sigma$. Here B is $\mathbb{P}^1$, Σ is a ruled rational surface and π_F has at least one section.

Now we concern ourselves with the question of how to go to the right boundary in the moduli space to remove the quantum corrections. First let us deal with Φ_{IIA}. Let T be a 2-torus in S_H which is a smoothly embedded elliptic curve for a suitable choice of complex structure on S_H. Let e_0 be a 2-form which is dual to the 2-cycle which is Poincaré dual to T. e_0 is a (1,1)-form for suitable complex structure on S_H. We may now expand the Kähler form on S_H in terms of a basis of 2-forms on S_H one of which is e_0. That is

$$J = \sum_i J_i e_i \,. \tag{2.1}$$

We claim that the limit $\Phi_{\text{IIA}} \to -\infty$ is equivalent to taking $J_0 \to \infty$.

This is much simpler to state in the case that S_H is an elliptic fibration with a section, $\pi_H : S_H \to B$. Now taking $\Phi_{\text{IIA}} \to -\infty$ is equivalent to taking the area of the section (or base) to be very large.

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This may be proven by using a fibre-wise duality argument much along the lines of [8]. In [8] it was argued that taking the heterotic string to be weakly-coupled is the same as taking the section of X as a K3-fibration to be very large. Here we are arguing the converse — the weakly-coupled type IIA string is dual to the heterotic string on an elliptically-fibred K3 surface with a big section. The fibre-wise duality argument of [8] works equally well in this case.

To proceed further we will assume that the heterotic K3 surface, S_H, is an elliptic fibration with a section. Note that this will reduce the number of moduli we are allowed to probe — not all K3 surfaces are elliptic with a section. Fortunately we will still be able to reach a boundary where all quantum corrections disappear.

We have argued above that this section of S_H is very large. If we assume that this K3 is generic (i.e., all fibres are of Kodaira type I_1) then we may take the volume of S_H to be very large by making sure that the elliptic fibres have large area.

Again we may let fibre-wise duality suggest an interpretation of such a limit in the type IIA picture. We follow an idea first explained in [4] following the work of [9]. Consider a dual pair of a heterotic string compactified on T^4 and a type IIA string compactified on a K3 surface. The map between the moduli describing these two compactifications is completely known [2]. We may thus ask what happens on the type IIA side if we take a 2-torus within the 4-torus on the heterotic side to have very large area. The result is that the K3 surface undergoes a "stable degeneration". Precisely what the stable degeneration is depends upon whether we wish to describe the $E_8 \times E_8$ heterotic string or the $\mathrm{Spin}(32)/\mathbb{Z}_2$ heterotic string on a large T^2. The $\mathrm{Spin}(32)/\mathbb{Z}_2$ case was described in [10]. We will focus on the $E_8 \times E_8$ case which was described in [4].

The result is that the K3 surface becomes a union of two rational elliptic surfaces intersecting along an elliptic curve. We will describe the geometry of the rational elliptic surface in more detail later. In terms of an elliptic fibration this union may be viewed as an elliptic fibration over two $\mathbb{P}^1$'s touching at a point. See [10, 11] for a more detailed description of this.

Now let us return to the case of a Calabi–Yau threefold, X, which is a K3-fibration. The type IIA string on this space is dual to the $E_8 \times E_8$ heterotic string on $S_H \times E_H$. In order to take the elliptic fibre of S_H to large area the above argument suggests that we should let each K3-fibre of X undergo the corresponding stable degeneration.

That is, X becomes a fibration over B where each fibre is now the union of two rational elliptic surfaces joining along an elliptic curve. We also want to view X as an elliptic fibration over a surface, Σ. In figure 1 we depict our stable degeneration in terms of this elliptic fibration. We need to introduce quite a lot of notation to describe various aspects of this degeneration.

Before the degeneration Σ is a ruled surface. To be specific, Σ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$. We depict these $\mathbb{P}^1$-fibres by vertical dotted lines in figure 1. We label a generic vertical line by f. In the simplest case we may assume that Σ is the Hirzebruch surface

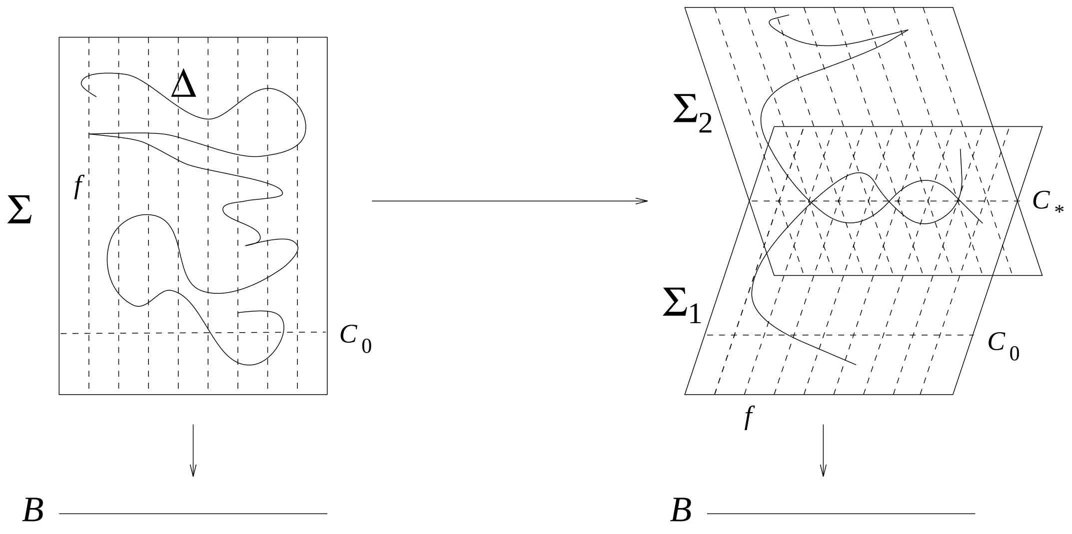


Figure 1: The stable degeneration for X.

$\mathbb{F}_n$ along the lines of [12, 9]. Viewing $\mathbb{F}_n$ as a $\mathbb{P}^1$-fibration, we can always find a section of self-intersection $-n$. We will denote a generic such section by C_0 and we also denote it by a dotted line in figure 1. Note that if $n > 0$ then this section is unique.

The elliptic fibration $\pi_F : X \to \Sigma$ is not smooth. Over a discriminant locus, $\Delta \subset \Sigma$, the fibres degenerate. We depict this locus by a solid line in figure 1.

Let $X^\sharp$ denote the degenerated version of X. When we go to the stable degeneration, every f-line will break into two lines intersecting at a point. Thus, the base of our elliptic fibration becomes a bundle over B with generic fibre given by *two* lines. Restricting the elliptic fibration to either of these two lines will give us a rational elliptic surface. Thus $X^\sharp$ is a fibration over B with fibre given by two rational elliptic surfaces intersecting along an elliptic curve. Equally, $X^\sharp$ is the union of two spaces, X_1 and X_2, intersecting along a surface which is an elliptic fibration. We denote this surface S_* (and thus $X^\sharp = X_1 \cup_{S_*} X_2$). Meanwhile Σ has become the union of two ruled surfaces, Σ_1 and Σ_2, intersecting along a $\mathbb{P}^1$. We denote this $\mathbb{P}^1$ by C_*. S_* is then the restriction of the elliptic fibration, $\pi_F^\sharp : X^\sharp \to (\Sigma_1 \cup_{C_*} \Sigma_2)$, to C_*.

One may show [10] that if Σ was given by $\mathbb{F}_n$ then each of Σ_1 and Σ_2 are isomorphic to $\mathbb{F}_n$ and C_* is a section of both Σ_1 and Σ_2. With respect to Σ_1 we may chose C_0 to be disjoint from the section C_*. In this case C_* becomes a section in the class of C_0 within Σ_2.

The discriminant locus will divide itself between Σ_1 and Σ_2. Note that if the discriminant within Σ_1 intersects C_* then the elliptic fibration of S_* will contain a bad fibre. For consistency the discriminant within Σ_2 must intersect C_* at the same point, and with the same degree.

2.2. The cohomology of the degeneration

Now that we know the way that X degenerates at the boundary of the moduli space we wish to consider how we can address the question as to how well the type IIA string (or F-theory) picture and the $E_8 \times E_8$ heterotic picture agree.

To do this we will focus on the Ramond-Ramond fields in the type IIA string which live in $H^{\mathrm{odd}}(X, \mathbb{R}/\mathbb{Z})$. Assuming $b_1(X) = 0$ this reduces to $H^3(X, \mathbb{R}/\mathbb{Z})$. Let us assume that $H_3(X)$ is torsion free.[4] This means that the Ramond-Ramond fields live in the "intermediate Jacobian" of X given by $H^3(X, \mathbb{R})/H^3(X, \mathbb{Z})$. We would therefore like to see how $H^3(X, \mathbb{Z})$ behaves as we perform the degeneration $X \to X^\sharp$.

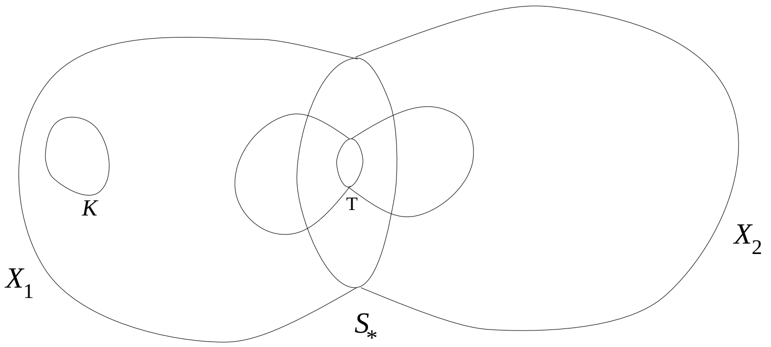


Figure 2: The stable degeneration $X^\sharp$.

Formally one may analyze this degeneration in terms of the Clemens-Schmid exact sequence [15]. Roughly speaking the following happens. Consider first a 3-cycle $K \in H_3(X_1)$ which does not intersect S_*. Clearly this should contribute to $H_3(X^\sharp)$. When counting these contributions the only worry might be that a 3-cycle in X_1 may be homologically the same as a 3-cycle in X_2. Actually this cannot happen since such an equivalence would imply the existence of an element of $H_3(S_*)$, but $H_3(\mathrm{K3}) = 0$. Thus $H_3(X_1) \oplus H_3(X_2) \subset H_3(X^\sharp)$.

The other contribution to $H_3(X^\sharp)$ arises in a more interesting way. Suppose we have a 2-cycle T in $H_2(S_*)$. There are natural maps $f_1 : H_2(S_*) \to H_2(X_1)$ and $f_2 : H_2(S_*) \to H_2(X_2)$. Let us suppose that T lies in the kernel of both of these maps. This means that S_* is the boundary of a 3-chain that lives in X_1 and another 3-chain that lives in X_2. Thus we may build an element of $H_3(X^\sharp)$ as shown in figure 2.

Let us assume that S_* is in the form of an generic elliptic fibration with a section. The elements of $H_2(S_*, \mathbb{Z})$ generated by the section and the generic fibre will not be in the kernel of f_1 and f_2 as they map to the section and fibre of X_1 and X_2. The other 20 two-cycles of S_* will be in the kernel. Let us denote this (self-dual) lattice of cycles by $M \cong \Gamma_8(-1) \oplus \Gamma_8(-1) \oplus U \oplus U$, where $\Gamma_8(-1)$ is the root lattice of E_8, taken with a negative signature, and U is the hyperbolic plane. M can be thought of as the lattice of cycles in S_* which are not algebraic. It is known as the "transcendental lattice". (Note that if S_* is not generic then we may expect fewer than 20 cycles to contribute

[4] If $H_3(X)$ contains torsion, the Ramond-Ramond moduli may include discrete degrees of freedom. This can lead to interesting problems in duality which are not fully understood at this time. These were discussed in [13, 14].

to the kernel and the transcendental lattice will fall in rank.) We have now argued that

$$H^3(X^\sharp, \mathbb{Z}) \cong H^3(X_1, \mathbb{Z}) \oplus H^3(X_2, \mathbb{Z}) \oplus M\,. \tag{2.2}$$

On the type IIA side therefore, our torus of Ramond-Ramond moduli has factorized itself nicely into a product of three tori. What do these three tori correspond to in the heterotic picture?

Firstly we have the heterotic K3 surface itself, S_H. We know the complex structure of S_H is given by the complex structure of S_*. Since we have taken the volume of S_H to be infinite it would be unreasonable to assume we have anything left of the Kähler form data of S_H. What remains then is the B-field. The obvious thing to do is to associate M with part of the moduli space of B-fields. This is natural since M was derived from $H^2(S_*, \mathbb{Z})$. Note that we only see the B-fields associated to the 20 transcendental cycles — we have "lost" two of the degrees of freedom in the B-field associated to the section and fibre of S_H.

Next we have the two E_8-bundles. As we shall see in the next section, and has been noted in [4, 16, 17], the remaining moduli of X_1 taken together with the torus $H^3(X_1, \mathbb{R})/H^3(X_1, \mathbb{Z})$ produces the hyperkähler moduli space of one E_8 bundle while the corresponding data on X_2 produces the other.

We have therefore achieved our goal. By going to this boundary in the moduli space of theories we have been able to identify the moduli space of theories exactly with the type IIA interpretation and exactly with the heterotic interpretation. This has been at a price of course. On the type IIA side we have lost some of the deformations of X and some of $H^3(X)$ — resulting in a loss of Ramond-Ramond moduli. On the heterotic string side we have lost the Kähler moduli of S_H and we have lost some deformations of complex structure by demanding that S_H be elliptic with a section. We also have only 20 of the original 22 B-field degrees of freedom. The boundary we have reached has real codimension 24.

3. The bundle

3.1. The Mordell–Weil group

In this section we will discuss the E_8-bundle moduli of the heterotic string in the type IIA language. The ideas we present in this section are essentially contained in Friedman, Morgan, and Witten's work [4] as well as [18, 16, 17]. We present here some explicit construction methods and we discuss some examples to show how easily the duality map can be seen. We also need to introduce an important lattice, Υ, for section 4. Our method will focus on the relationship between the associated spectral curve of the bundle and the Mordell–Weil group of rational elliptic surfaces.

X_1 is an elliptic fibration over $\Sigma_1 \cong \mathbb{F}_n$. Deformations of this threefold, together with its intermediate Jacobian, will give the moduli space of one of the E_8-bundles. Again we denote by C_0 a section of $\mathbb{F}_n$ with self-intersection $-n$. C_* is another section, disjoint from C_0, with self-intersection $+n$. We denote a generic $\mathbb{P}^1$-fibre of $\mathbb{F}_n$ by f. We thus have the intersection relations $[C_0].[C_0] = -n$, $[C_0].[f] = 1$, and $[f].[f] = 0$.

Now write X_1 in Weierstrass form

$$y^2 = x^3 + ax + b\,, \tag{3.1}$$

where a and b are sections of line bundles over Σ_1. Let A and B denote the divisor classes corresponding to these bundles. The discriminant is given by $4a^3 + 27b^2$. Let us denote the class of the discriminant by Δ. Sometimes we will be sloppy with our notation and refer to the discriminant locus itself by Δ.

One may show that [10]

$$\begin{aligned} [C_*] &= [C_0] + n[f] \\ A &= 4[C_0] + 8[f] \\ B &= 6[C_0] + 12[f] \\ \Delta &= 12[C_0] + 24[f]\,. \end{aligned} \tag{3.2}$$

In the generic case, Δ collides with f transversely at 12 points. This means that, as expected, we obtain an rational elliptic surface by restricting the elliptic fibration to f. In Kodaira language this corresponds to building a rational elliptic surface by an elliptic fibration with twelve I_1 fibres.

There are many ways of building a rational elliptic surface in Weierstrass form. Indeed, Persson has listed all 289 ways explicitly [19]. Of central importance to us will be the notion of the Mordell–Weil group of this surface — i.e., the group of sections taking our given section as the identity. The Mordell–Weil group of all the possible rational elliptic surfaces was also determined in [19].

One may show that the homology of 2-cycles of the rational elliptic surface naturally decomposes into three parts:

1. The identity section and the fibre.

2. Components of fibres not touching the identity section.

3. Elements generated from the Mordell–Weil group.

The Picard lattice of the rational elliptic surface is isomorphic to $\Gamma_8(-1) \oplus U$ of which the first part accounts for U. This, the components of fibres not touching the 0-section and the Mordell–Weil group must account for $\Gamma_8(-1)$. Components of fibres missing the 0-section arise when we have bad fibres in the Kodaira classification other than I_1 and II. These are the two-cycles which are shrunk down to zero size to generate enhanced gauge symmetries in the F-theory limit as in [12]. Alternatively in type IIA language, these are the two-cycles which are shrunk down to zero size in order to switch off the Wilson lines around the heterotic 2-torus, E_H, again generating an enhanced gauge symmetry.

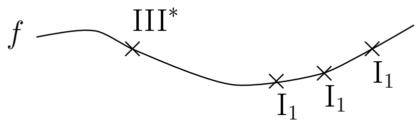
Since these 2-cycles which miss the 0-section are associated to (perturbative) enhanced gauge symmetry, their existence must correspond to the actual structure group of our supposed E_8-bundle actually being less than we thought. To be precise, let the actual structure group of the bundle be $\mathcal{G}$. Then the observed gauge symmetry is given by the centralizer of $\mathcal{G} \subset E_8$. Note that the Mordell–Weil contribution towards H_2 is the complement of this enhanced gauge symmetry contribution within $\Gamma_8(-1)$. This tells us that the Mordell–Weil group part of the contribution must be closely associated to $\mathcal{G}$. Indeed we shall now see how the Mordell–Weil group of each rational elliptic surface is key in constructing the heterotic bundles.

In the generic case we have twelve I_1 fibres and no enhanced gauge symmetry. In this case the Mordell–Weil group is rank 8 and the structure group really is $\mathcal{G} \cong E_8$. As this is the hardest to visualize let us try something a little more manageable.

At the other extreme of the bundle with a structure group given by the complete E_8, we may try to build a bundle with a trivial structure group. This may be done and corresponds to "point-like instantons". This case is somewhat subtle and we postpone it for a while.

3.2. SU(2)-bundles

Instead we try to build an SU(2)-bundle. Since $\mathfrak{e}_7 \oplus \mathfrak{sl}(2)$ is a maximal subalgebra of $\mathfrak{e}_8$, the centralizer of SU(2)$\subset E_8$ is E_7.[5] Thus if the heterotic string is compactified on an SU(2)-bundle, there will be an unbroken E_7 gauge symmetry. The rational elliptic curve we desire as our generic fibre over a point in B is given by



(3.3)

This has Mordell-Weil group isomorphic to $\mathbb{Z}$, i.e., of rank 1. This means that there is another section, σ_1, of this rational elliptic surface which generates the Mordell–Weil group. The difference between this section and our zero section, σ_0, generates part of H_2 of the surface (see, for example, [20]), $[\sigma_1 - \sigma_0] \in H_2$.

Remember that we are building X_1 as a rational elliptic surface fibration over B. There can be monodromy in this fibration such that our 2-cycle $[\sigma_1 - \sigma_0]$ is not an invariant homology cycle. It is possible that parallel transport along a closed loop in B may map $[\sigma_1 - \sigma_0]$ to $-[\sigma_1 - \sigma_0] = [\sigma_{-1} - \sigma_0]$, where σ_{-1} is another section in the Mordell–Weil group. Under the group law, σ_{-1} is the inverse of σ_1.

Now this rational elliptic surface intersects S_* along an elliptic curve. This elliptic curve is the elliptic fibre over the point in Σ_1 given by the collision of C_* with the f-line we have taken to build our rational elliptic surface. The sections σ_1 and σ_{-1} will each intersect this elliptic curve at two points, P_1 and P_{-1}. Consider now the whole family

[5] Note that $E_7\times$SU(2) is *not* a subgroup of E_8 however!

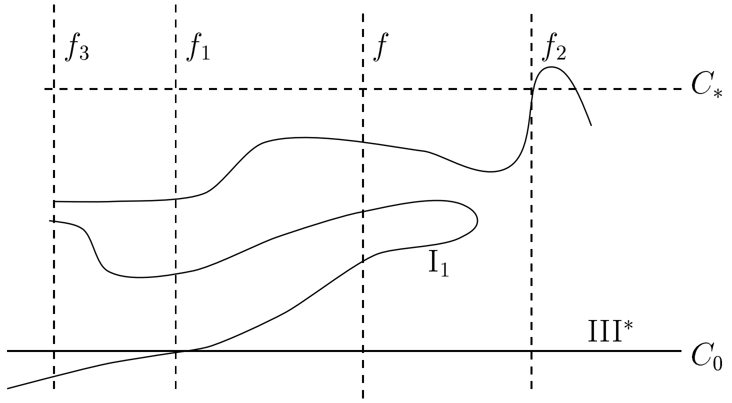


Figure 3: The discriminant for an SU(2)-bundle.

of rational elliptic surfaces over B. Transporting P_1 and P_{-1} around the whole of B will build a double cover $\pi_s : C \to B$. This curve, C, is the *spectral curve.*

Globally, in terms of Σ_1, our picture for this elliptic fibration looks like figure 3. Over the generic f-line, labelled by f in the figure, we have one III^* fibre and three I_1 fibres as desired for our generic rational elliptic surface. In the generic case there are three things that can happen to spoil this for some particular f-lines. In the following recall that n refers to the geometry of $\Sigma_1 \cong \mathbb{F}_n$.

(a) The curve of I_1's may collide with the line C_0 of III^* fibres. This does not occur transversely — each collision has multiplicity 3. One may also show from intersection theory that such a collision occurs $8 - n$ times [9]. When the elliptic fibration is restricted to an f-line through this collision (labelled f_1 in the figure) we obtain a rational elliptic surface with one II^* fibre and two I_1 fibres.

(b) The curve of I_1's may be tangential to the f direction. When the elliptic fibration is restricted to such an f-line (labelled f_2 in the figure) we obtain a rational elliptic surface with one III^* fibre, one I_2 fibre and one I_1 fibre.

(c) The curve of I_1's may have a cusp. An f-line passing through such a point is labelled by f_3 in the diagram. Now the resulting rational elliptic surface has one III^* fibre, one II fibre and one I_1 fibre.

In the case (a), the Mordell–Weil group becomes trivial. This must mean that the sections σ_1 and σ_{-1} must coincide with σ_0 at this point over B. In case (b), the Mordell–Weil group becomes $\mathbb{Z}_2$. For this surface both σ_1 and σ_{-1} pass through a 2-torsion point in the elliptic fibre over C_*. In both of these cases, σ_1 and σ_{-1} coincide and we have a branch point of $\pi_s : C \to B$. In case (c), the Mordell–Weil group remains equal to $\mathbb{Z}$ and $\pi_s : C \to B$ is not branched.

We may calculate the genus of C from the number of branch points. As mentioned earlier, the are $8 - n$ collisions from case (a). To count case (c) we look for collisions between A and B given by (3.2). For a similar computation see the computation in section 6.6 of [2]. The result is $20 - n$.

To calculate (b) we need to write our discriminant more explicitly. Let s and t be affine coordinates on $\Sigma_1 \cong \mathbb{F}_n$. We let s be a coordinate in the "fibre" direction and t be in the "base" direction. The curve of I_1 fibres in figure 3 is in the class $\Delta - 9[C_0] = 3[C_0] + 24[f]$. This means that the polynomial, δ_1, whose zero gives the curve of I_1 fibres can be written in the form

$$\delta_1 = s^3 f_{24}(t) + s^2 f_{24-n}(t) + s f_{24-2n}(t) + f_{24-3n}(t)\,, \tag{3.4}$$

where $f_m(t)$ represents some generic polynomial of degree m in t. Similar equations appeared in [9]. The discriminant of δ_1 with respect to s is a polynomial of degree $96 - 6n$ in t. This measures the number of points over the base where the three solutions of the cubic equation given by (3.4) coincide. This occurs for cases (b) and (c).

It is easy to show that a generic cusp of type (c) will contribute 3 towards the above discriminant while the tangency of type (b) will contribute only 1. Thus the number of occurrences of case (b) is $96 - 6n - 3(20 - n) = 36 - 3n$. The number of branch points is therefore given by $(8 - n) + (36 - 3n) = 44 - 4n$ from which is it follows that C has genus $\frac{1}{2}(44 - 4n) - 1 = 21 - 2n$.

We wish to claim that there is a natural identification of the Jacobian of C, $H^1(C, \mathbb{R}/\mathbb{Z})$, with the moduli space of Ramond-Ramond fields $H^3(X_1, \mathbb{R}/\mathbb{Z})$. This is easy given our construction of the spectral curve. First let us ask how we see $H_3(X_1)$ in terms of the fibration $p_1 : X_1 \to B$. Such 3-cycles are built from transporting 2-cycles within fibre around non-contractable 1-cycles in B. These 2-cycles must be non-monodromy-invariant to build a homologically nontrivial element of $H_3(X_1)$. These 2-cycles come exactly from the Mordell–Weil group of the fibre of p_1. We may now ask how we build elements of $H_1(C)$ in terms of the fibration $\pi_s : C \to B$. The answer is very similar except now we transport the points in the fibre around loops in B. By construction however we identify points in the fibre of π_s with the 2-cycles coming from the Mordell–Weil group in the fibre of p_1. Thus $H_1(S) \cong H_3(X_1)$ and so $H^1(C, \mathbb{R}/\mathbb{Z}) \cong H^3(X_1, \mathbb{R}/\mathbb{Z})$.

Actually we may do a little better than this. The Jacobian of a curve is not only a torus but an *abelian variety*. That is, it is a torus which admits a complex structure and can be embedded in some complex projective space. One can argue that the same is true for the intermediate Jacobian for X_1. One can then show that the Jacobian of C, $\mathrm{Jac}(C)$, is isomorphic to the intermediate Jacobian of X_1 as an algebraic variety (for example, see the Abel-Jacobi map of [21]).

We may also phrase our construction more formally by using the Leray spectral sequence of a fibration and noting that $H^1(B, \pi_{s*}\mathbb{Z}) \cong H^1(B, R^2 p_{1*}\mathbb{Z})$.

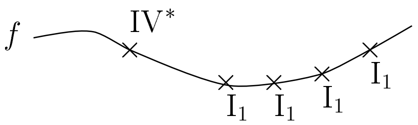
We are done once we note that an SU(2)-bundle on S_H is specified uniquely by a given spectral curve, $C \subset S_H$, and a line bundle (of a particular degree) $L \to C$. This was shown in [4]. The deformations of the spectral curve are given by the deformations of X_1 and the moduli space of the line bundle L are given by the Jacobian of C and thus the Ramond-Ramond fields.

As a final check let us compute the dimensions of the moduli space. The moduli space of SU(2)-bundles on a K3 surface is a hyperkähler space whose quaternionic dimension is given by Hirzebruch-Riemann-Roch as $2c_2 - 3$. Similarly the moduli space of L together with its Jacobian forms a hyperkähler space as in Hitchin's construction [22]. The quaternionic dimension of this moduli space is given by the genus of L which is $21 - 2n$. Thus we are in agreement if $c_2 = 12 - n$ as expected [12].

3.3. G_2-bundles

Having explicitly mapped the moduli space of SU(2)-bundles on S_H to its type IIA dual picture in terms of X_1, we may try to do the same for a larger structure group, $\mathcal{G}$. The obvious thing to do is to slightly relax our constraint above that we have an unbroken E_7 gauge symmetry.

As the next case we use the generic rational elliptic surface in the fibre of $p_1 : X_1 \rightarrow B$ of the following form:



(3.5)

The type IV* fibre is associated to an E_6 singularity and so one might at first suppose that we are going to see an unbroken E_6 gauge symmetry. This is not the case however thanks to monodromy [23]. In fact there is more than one way in which this case will be quite a bit more subtle than the SU(2) case above.

In figure 4 we draw the discriminant locus of the elliptic fibration $X_1 \rightarrow \Sigma_1$ again. Now we have the following collisions:

(a) The curve of I_1's may collide with the line C_0 of IV* fibres. This does not occur transversely — each collision has multiplicity 2. One may also show from intersection theory that such a collision occurs $2(6-n)$ times. When the elliptic fibration is restricted to an f-line through this collision (labelled f_1 in the figure) we obtain a rational elliptic surface with one III* fibre and three I_1 fibres.

(b) The curve of I_1's may be tangential to the f direction. When the elliptic fibration is restricted to such an f-line (labelled f_2 in the figure) we obtain a rational elliptic surface with one IV* fibre, one I_2 fibre and two I_1 fibres.

(c) The curve of I_1's may have a cusp. An f-line passing through such a point is labelled by f_3 in the diagram. Now the resulting rational elliptic surface has one IV* fibre, one II fibre and two I_1 fibres.

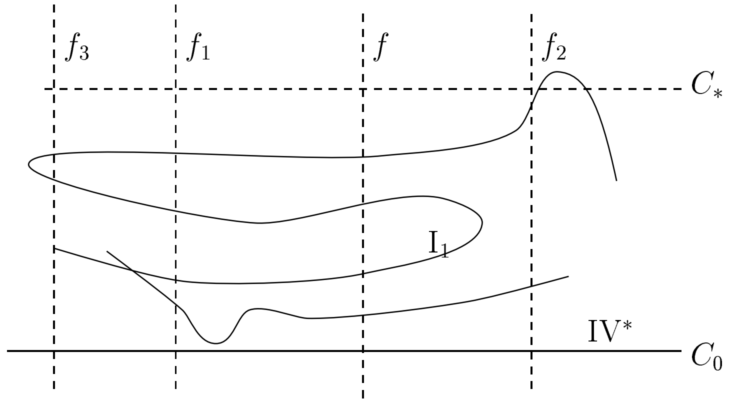


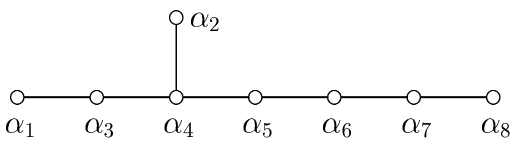
Figure 4: The discriminant for a G_2-bundle.

Over a generic f, the rational elliptic surface has a Mordell–Weil group equal to $\mathbb{Z} \oplus \mathbb{Z}$. Over f_1 and f_2 in cases (a) and (b) the Mordell–Weil group becomes $\mathbb{Z}$. Thus we again have some kind of branched curve over B with branch points at the locations of f_1 and f_2.

What kind of monodromy will we get for these two-cycles coming from the Mordell–Weil group? We may address this question from our knowledge that the total lattice of cycles generated by the Mordell–Weil group and the reducible fibres is $\Gamma_8(-1)$ as explained earlier. Over the generic curve f in figure 4 we have the type IV^* fibre which contributes an $\mathfrak{e}_6$ root lattice of cycles to the Picard lattice. Thus the Mordell–Weil group must contribute the orthogonal complement of this $\mathfrak{e}_6$ root lattice which is the $\mathfrak{sl}(3)$ root lattice. The monodromy around the f_2 fibre in figure 4 leaves the $\mathfrak{e}_6$ root lattice alone and so we may obtain an element of the Weyl group of $\mathfrak{sl}(3)$ acting on the Mordell–Weil part.

The monodromy around the f_1 line is more interesting. It is known that the monodromy around this line is not trivial on the IV^* fibre and thus not trivial on the $\mathfrak{e}_6$ root lattice. Actually the monodromy acts as an outer automorphism of $\mathfrak{e}_6$. Since the full $\Gamma_8(-1)$ lattice of cycles must be mapped to itself by this monodromy we need to know how such an outer automorphism acts within the context of the $\mathfrak{e}_8$ root lattice.

Let us label the simple roots of $\mathfrak{e}_8$ as follows:

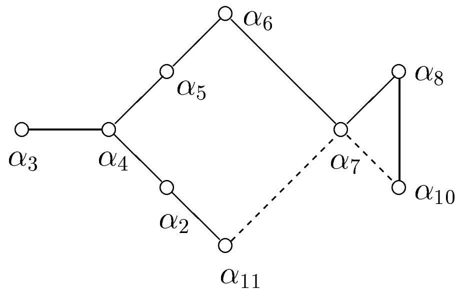


(3.6)

Now introduce some more roots

$$
\begin{aligned}
\alpha_9 &= -2\alpha_1 - 3\alpha_2 - 4\alpha_3 - 6\alpha_4 - 5\alpha_5 - 4\alpha_6 - 3\alpha_7 - 2\alpha_8 \\
\alpha_{10} &= -\alpha_8 - \alpha_9 \\
\alpha_{11} &= -\alpha_1 - 2\alpha_2 - 2\alpha_3 - 3\alpha_4 - 2\alpha_5 - \alpha_6 \,.
\end{aligned}
\tag{3.7}
$$

We may draw a Dynkin-like diagram representing the angles between some of these roots as follows:



(3.8)

Here a dotted line indicates that the inner product is negative that of a solid line. Clearly this diagram shows that we have a symmetry of the root system by reflecting (3.8) about a horizontal line and taking $\alpha_7 \to -\alpha_7$.

This symmetry of the root system is an element of the Weyl group of $\mathfrak{e}_8$. The subdiagram to the left of α_7 in (3.8) is the Dynkin diagram of $\mathfrak{e}_6$ and this symmetry corresponds to the outer automorphism. The subdiagram to the right of α_7 in (3.8) is the Dynkin diagram of $\mathfrak{sl}(3)$. It must be then that the monodromy around f_1 exchanges the two simple roots of $\mathfrak{sl}(3)$.

This is not in the Weyl group of $\mathfrak{sl}(3)$. It is in the Weyl group of $\mathfrak{g}_2$ however. Indeed, if $W(\mathfrak{h})$ is the Weyl group of the Lie algebra $\mathfrak{h}$ then

$$0 \to W(\mathfrak{sl}(3)) \to W(\mathfrak{g}_2) \to \mathbb{Z}_2 \to 0\,, \tag{3.9}$$

where this latter $\mathbb{Z}_2$ is precisely the exchange generated above by the monodromy.

From the heterotic language we can argue that the bundle we are constructing here is indeed a G_2-bundle and not an SU(3)-bundle. This is because the monodromy causes the effective gauge group to be observed to be F_4 rather than E_6 [23]. Since there is a maximal subgroup

$$E_8 \supset G_2 \times F_4\,, \tag{3.10}$$

we should expect an F_4 gauge symmetry to be preserved by a G_2-bundle. Indeed the way that one can demonstrate the maximal embedding of (3.10) is via an argument along the lines of the diagram in (3.8).

This example shows a rule (which follows from Donagi's construction [24]) that the Weyl group of the structure group, $\mathcal{G}$, of the bundle is generated by the monodromy on the Mordell–Weil part of the Picard lattice of the rational elliptic surfaces.

We also see that if the $2(6-n)$ collisions of the type along f_1 in figure 4 coalesce into pairs then we lose any monodromy of this type and reduce the monodromy group to that of SU(3). This is in agreement with the similar statement that such a pairing results in an E_6 unbroken gauge symmetry.

Take one of the generators of the Mordell-Weil group, $\mathbb{Z}\oplus\mathbb{Z}$, of the rational elliptic surface in a generic fibre. This has an orbit of six elements via the monodromy around

B. We may therefore build a six-fold branched cover $\pi_s : C \to B$. This again we may call the "spectral curve" as in section 3.2. It is important to note however that this spectral cover for the G_2 case is not quite as simple as that for the SU(2) case.

It is now untrue that $H^1(B, \pi_{s*}\mathbb{Z}) \cong H^1(B, R^2 p_{1*}\mathbb{Z})$ and so we cannot identify the Ramond-Ramond moduli of the type IIA string with the Jacobian of the spectral curve. This story is familiar one in the theory of branched covers and has been studied extensively by Donagi in the current context [24].

The general idea is that one may identify a "Prym" which is a subspace of the Jacobian of the spectral curve. This Prym is also an abelian variety. One can then show that this Prym matches the Ramond-Ramond moduli in terms of the type IIA theory and that it correctly reproduces the moduli space of the bundle on the heterotic side. The former was shown by Kanev [25] and the latter by Donagi [26].

The basic construction of the Prym is as follows. Let W be the Weyl group of $\mathcal{G}$. The spectral curve may be thought of as an R-cover of B where R is some representation of W. That is to say, the point-like fibres of the map $\pi_s : C \to B$ form a representation of W under monodromy. The problem with G_2 that we did not have with SU(2) in the last section, is that to find the agreement $H^1(B, \pi_{s*}\mathbb{Z}) \cong H^1(B, R^2 p_{1*}\mathbb{Z})$ we have the *wrong* representation. In order to find agreement, we need the natural representation of the Weyl group on the root space — i.e., the representation with dimension equal to $\mathrm{rank}(\mathcal{G})$.

Thus we replace one cover of B with another one and try to take the latter's Jacobian. We cannot do this literally as the root space representation of the Weyl group is not a permutation representation. Thus the supposed spectral curve for this representation cannot actually be thought of as a branched cover. Instead consider the "cameral cover". This is a (W-Galois) cover where the representation is given by Weyl chambers. This is a d-fold cover of B where d is the number of elements in G. Let $\widetilde{C}$ be the cameral cover. Let Λ be the root lattice of $\mathcal{G}$ and let $\mathrm{Jac}(\widetilde{C})$ be the Jacobian of $\widetilde{C}$. Clearly W has an action on both Λ and $\mathrm{Jac}(\widetilde{C})$. The Prym we desire may be written as

$$P = \mathrm{Hom}_W(\Lambda, \mathrm{Jac}(\widetilde{C}))\,, \tag{3.11}$$

where the W subscript means that $p(g\lambda) = gp(\lambda)$ for all $p \in P$, $g \in W$, $\lambda \in \Lambda$. In general this is a disconnected set of abelian varieties. It is an interesting question how to deal with the cases where there is more than one component in this Prym. Here we will assume we are dealing with the component connected to the identity if there is more than one component. In other words there is an l such that

$$P = \mathbb{C}^l/\Upsilon\,, \tag{3.12}$$

where Υ is a lattice of dimension $2l$. A little algebra shows that

$$\Upsilon = \mathrm{Hom}_W(\Lambda, H^1(\widetilde{C}, \mathbb{Z}))\,. \tag{3.13}$$

In the simplest case of $\mathcal{G} \cong \mathrm{SU}(N)$, the Weyl group is the symmetric group on N objects, the spectral curve is an N-fold cover of B, the Prym is equal to the Jacobian of the spectral cover[6] and $\Upsilon = H^1(C, \mathbb{Z})$. For general $\mathcal{G}$ life is more complicated.

If the situation is sufficiently generic that each monodromy corresponds to a Weyl reflection in one root then l may be determined. Let there be b branch points. We show in the appendix that

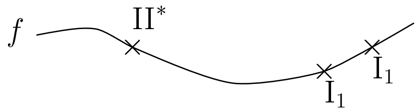
$$l = \tfrac{1}{2}b - \mathrm{rank}(\mathcal{G})\,. \tag{3.14}$$

Applying this formula to our G_2-bundle, we have $2(6-n)$ collisions of type (a) and it is straight-forward to show that there are $60-6n$ collisions of type (b). Thus $b = 72-8n$ and $l = 34-4n$. This value of l agrees with the quaternionic dimension of the moduli space of G_2-bundles on a K3 with $c_2 = 12-n$ again.

It is a simple matter now to extend this construction to $\mathcal{G} = \mathrm{SU}(3)$. If we allow the collisions of type (a) to pair up, we lose the monodromy which elevated SU(3) to G_2. Thus we should only have an SU(3)-bundle. Now we only count the type (b) collisions as branch points and so $b = 60-6n$ and $l = 28-3n$ again in agreement with the direct bundle computation.

3.4. Bundles with no structure group

Now we want to try to build a bundle with no structure group at all. This suggests picking the generic rational elliptic surface in X_1 to have a trivial Mordell–Weil group. This forces the rational elliptic surface to be of the form



(3.15)

or by letting the two I_1's coalesce to form a type II fibre.

If every f is of this form within Σ_1 then we will have a whole curve of type II* fibres. By the usual F-theory recipe this means we have an unbroken E_8 gauge symmetry. This is most reasonable if our bundle has no structure group! Of course, what we are talking about here are the "point-like instantons" of [27, 12].

The geometry of $E_8 \times E_8$ point-like instantons in F-theory or type IIA language has been explained in great length in various places (for example, [9, 2]). The key point is that within Σ_1, there are collisions between the curve of II* fibres and the curve of remaining I_1 fibres within Δ. A simple calculation in intersection theory shows this happens at $12-n$ points. The resulting threefold, X_1, produced by this fibration is highly singular over each of these $12-n$ points. It may be smoothed out by blowing

[6] Actually we should be more careful and say there exists a spectral curve such that the Jacobian of this curve is isomorphic to the Prym. The construction from the Mordell–Weil group produces the spectral curve from the permutation action of W on the roots of $\mathcal{G}$ rather than the N-dimensional permutation representation. These two spectral curves happened to coincide for $\mathcal{G} \cong \mathrm{SU}(2)$.

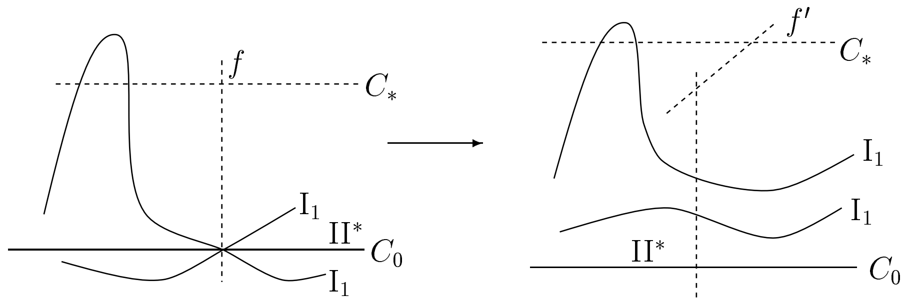


Figure 5: The blow-up for a generic point-like E_8 instanton.

up each such collision in Σ_1. Each such collision and the resulting blow-up in shown in figure 5. In F-theory language these blow-ups give tensor multiplets. Each of the $12-n$ collisions is associated to a point-like instanton for the heterotic string on a K3 surface where each instanton has $c_2 = 1$. Thus we have a total c_2 given by $12-n$ yet again.

What we would like to consider now is whether we actually have the correct moduli space for these $12-n$ points. Each point-like instanton should presumably have as its moduli space the K3 surface, S_H, itself. We may restrict attention to such a single instanton by fixing $n = 11$.

To achieve the configuration of the discriminant locus in figure 5 we are required to restrict the forms of the polynomials a and b in equation (3.1). A type II* fibre requires a to vanish order at least 4 and b to vanish order 5. The divisors class B therefore divides into two parts $B = B' + 5[C_0]$, where $B' = [C_0] + 12[f]$. The collisions in figure 5 occur at collisions between B' and C_0. We are free to move these collisions around by varying b.

It was argued in [10] that the location of these collisions corresponded to the point-like instantons. To be more precise, consider the elliptic fibration of $S_H \cong S_*$ given by $\pi_H : S_H \to B$. We also have the fibration $p : \Sigma_1 \to B$. If a point-like instanton lives at a point $x \in S_H$ and the collision in the left of figure 5 occurs at a point $y \in \Sigma_1$ then $\pi_H(x) = p(y)$. Thus by varying the complex structure of X_1 we may move the point-like instanton around in the "base" direction of S_H.

In order to show that the moduli space of this point-like instanton is given by S_H we need to be able to vary its position in the "fibre" direction of S_H. This degree of freedom is provided by the Ramond-Ramond degrees of freedom over X_1 as we now show.

On the right-hand side of figure 5 the dotted line f' denotes the proper transform of the f-curve which had passed through the collision point on the left-hand side. As can be seen from the figure, f' does not touch the discriminant locus of the resulting elliptic fibration. Thus, the elliptic fibration restricted to f' has no bad fibres. This can only happen if the fibre is constant. Let us refer to this constant elliptic curve as Q. Thus $Q \times f' \subset X_1$.

Wedging by the generator of $H^2(f', \mathbb{Z})$ gives an injective map $H^1(Q, \mathbb{Z}) \to H^3(X_1, \mathbb{Z})$.

Thus Q contributes 2 real degrees of freedom to the Ramond-Ramond moduli.

We may rephrase this in terms of spectral curves again. As the Mordell–Weil group of our generic rational elliptic surface is now trivial, one might at first think that the spectral curve has simply collapsed to the zero section of $\pi_H : S_H \to B$. Let us instead say that the spectral curve, C, has degenerated in this case and has become reducible, $C = B \cup Q$. Now the Jacobian of the spectral curve is isomorphic to the Jacobian of Q. One can map this Jacobian into the intermediate Jacobian of X_1. Again, this is essentially the Abel-Jacobi mapping of [21].

Now let us construct the moduli space of one instanton, $\mathscr{M}_1$. As argued above, there is a fibration $\mathscr{M}_1 \to B$ where the degree of freedom in B is generated by moving around the point of collision of Δ within C_0. The fibre of this map is $\mathrm{Jac}(Q)$. However, since Q is constant all along f', Q is actually the elliptic fibre of $\pi_H : S_H \to B$ over the same point in B. Thus we construct $\mathscr{M}_1$ by replacing each elliptic fibre in $\pi_H : S_H \to B$ by its Jacobian. Given that π_H has a section, one may show that the resulting fibration has the same complex structure as the original (for example, see proposition 5.3.2 of [28]). *Thus we see that $\mathscr{M}_1 \cong S_H$ as desired.*

Actually this fibration of $\mathscr{M}_1$ is a little more subtle than first meets the eye. We have assumed that Q is a smooth elliptic curve. That is, we have assumed that our point-like instanton lives in a smooth fibre of $\pi_H : S_H \to B$. It is quite possible however that it lives in a bad fibre, such as an I_1 fibre. In this case the story of the discriminant locus is given in figure 6.

In this case Q is a degenerate elliptic curve as one might expect. What is interesting is how we now have a transverse intersection of two curves of I_1 fibres in the discriminant. Such a collision produces a singularity in X_1 which generically cannot be resolved. That is, X_1 *degenerates when we move the point-like instanton onto a bad fibre of S_H.* This is the only way that we could have $\mathscr{M}_1 \cong S_H$. When we are on a bad fibre, the intermediate Jacobian must degenerate and so X_1 must degenerate.

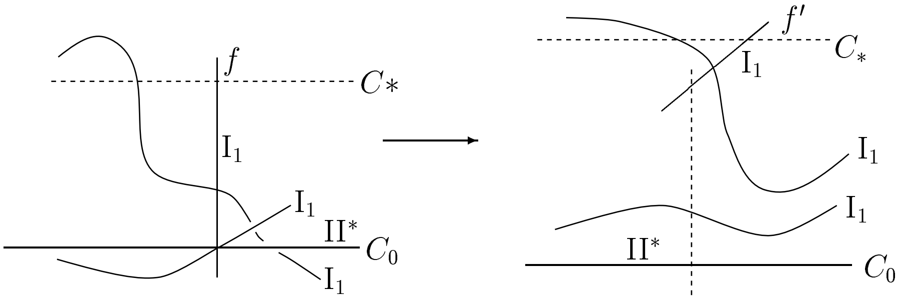


Figure 6: The blow-up for a point-like E_8 instanton on a bad fibre.

For more than one instanton we let

$$C = B \cup \bigcup_{i=1}^{12-n} Q_i \,, \tag{3.16}$$

where the Q_i are the constant elliptic curves coming from the scrolls formed by each of the blow-ups required. It is not hard to see that our construction above will produce a Hilbert scheme of $12-n$ points in S_H as expected.

In the language above, our Prym is the product of the Jacobians of Q_i and so

$$\Upsilon = \bigoplus_{i=1}^{12-n} H^1(Q_i, \mathbb{Z})\,. \tag{3.17}$$

It is known that there is a close relationship between point-like instantons on bad fibres and the very subtle case where the bundle on S_H in question is the tangent bundle. This has been discussed in [29]. It was also explicitly seen in [30] in terms of a construction from toric geometry that the spectral curve degenerated in precisely the way we have described here in the tangent bundle case.

4. Discussion

Let Υ_1 and Υ_2 be the lattices introduced in the last section from the two E_8-bundles or sub-bundles thereof and let M be the transcendental lattice of the K3 on which the heterotic string is compactified. Let $H^3(X,\mathbb{Z})_0 \cong H^3(X^\sharp,\mathbb{Z})$ be the monodromy-invariant part of $H^3(X,\mathbb{Z})$ as we go around the stable degeneration discussed in section 2.

The boundary of the moduli space corresponding to this degeneration of either the heterotic string or the type IIA string is of the following form. It acquires a fibration structure $\pi : \mathscr{M} \to B_{\mathscr{M}}$ where the complex dimension of $B_{\mathscr{M}}$ equals the quaternionic dimension of $\mathscr{M}$ and the generic fibre is an abelian variety $\mathbb{C}^p/L$ for some p and some lattice L. In identifying the moduli spaces, we must identify these abelian varieties and thus the description of the lattice L in the heterotic picture and the type IIA picture.

This leads to the key claim that

$$H^3(X,\mathbb{Z})_0 \cong M \oplus \Upsilon_1 \oplus \Upsilon_2\,. \tag{4.1}$$

This equation is essential to the notion of duality between the heterotic string on a K3 surface and the type IIA string (or F-theory) on a Calabi–Yau threefold.

As well as automatically identifying the abelian fibres of the moduli space, (4.1) also indicates how to naturally identify the bases, $B_{\mathscr{M}}$, as follows.[7] Recall that in the type IIA string $B_{\mathscr{M}}$ represents the moduli space of complex structures on X and in the heterotic string $B_{\mathscr{M}}$ represents the moduli space of complex structures on S_H and the moduli of the spectral curve for the bundle. This map shows how the variation of Hodge structure on X, in terms of periods of the holomorphic 3-form on elements of $H^3(X,\mathbb{Z})_0$, may be reduced into a statement about the variation of Hodge structure on

[7]This argument only really shows how the Heterotic picture of $B_{\mathscr{M}}$ and the type IIA picture of $B_{\mathscr{M}}$ can be identified locally. It is possible that type IIA $B_{\mathscr{M}}$ may map many-to-one to the heterotic $B_{\mathscr{M}}$. It is reasonable to expect this to happen when the Prym has more than one component.

S_H, in terms of periods of the holomorphic 2-form on this K3 surface on elements of the transcendental lattice, M. The bundle data becomes encoded in a variation of Hodge structure of a one dimensional object — the spectral curve. In the case of $\mathcal{G} \cong \mathrm{SU}(N)$, Υ is simply $H^1(C,\mathbb{Z})$ and we recover the full moduli space of C. In general we only have a sub-variation of Hodge structure for the spectral curve.

Thus the duality between the heterotic string and the type IIA string appears to be encapsulated by relating the 3-dimensional structure of the type IIA compactification to a 2-dimensional and a 1-dimensional structure for the heterotic string. That is, the Calabi–Yau threefold is related to a K3 surface and a bundle.

It is interesting to compare this situation to mirror symmetry. One of the simplest ways of describing the mirror symmetry principal for a Calabi–Yau threefold is the exchange of the even (or vertical) integral cohomology with the odd (or horizontal) integral cohomology. This picture was first suggested in the work of [31, 32] and then spelt out more clearly in section 5 of [33] and [34].[8]

The way that these integral cohomologies can be identified requires one to go to the "large complex structure" of one of the Calabi–Yau threefolds. This was explained in [35]. Only at this boundary *point* in the moduli space did one expect the two mirror theories to be exactly equivalent. Away from this boundary point the mirror equivalence becomes complicated by world-sheet instanton effects which come from rational curves.

Heterotic/type IIA duality has a much richer structure than mirror symmetry. Firstly our boundary where the two theories agree exactly is no longer a point but has a fairly large dimension. Secondly one can see that *both* the heterotic string and the type IIA string are prone to instanton effects. Each of these can be probed individually by moving away from the boundary in a certain direction.

For example, in order to probe the heterotic string world-sheet instanton effects[9] we need to move away from the boundary without changing the dilaton in the type IIA string. In effect we need to keep the section of S_H infinitely large but we may let the elliptic fibres become finite in size. Clearly this moves us away from the stable degeneration discussed in section 2. It is also evident that the exact correspondence between our moduli spaces will break down. Indeed we may go around a closed loop in the moduli space of X producing a highly non-trivial monodromy on $H^3(X,\mathbb{Z})_0$. If there were no quantum corrections to this moduli space then such a monodromy would have to correspond to some sort of T-duality on the moduli space of heterotic strings. This would imply that the moduli space of complex structures on X would be globally *exactly* in the form of some Teichmüller space divided out by this T-duality group. This is not true for a generic Calabi–Yau threefold. Thus the periods in our heterotic string on a K3 surface must generically become "mixed" between the variation of Hodge

[8]Actually this definition of mirror symmetry has yet to be shown to be consistent with other definitions. In a few examples where the integral cohomology is calculable and the rational curve can be counted, it does appear to work.

[9]There may also be higher-loop perturbative corrections in α'.

structure of S_H and the variation of Hodge structure of the spectral curve. That is, the notion of a K3 surface and the notion of a bundle will become somewhat confused. This is analogous to the way that the notion of a 0-cycle and a 2-cycle can be confused by mirror symmetry away from the large radius limit.

This confusion in the heterotic string is presumably accounted for by some world-sheet instanton effect just as rational curves appeared in mirror symmetry. It would be very interesting to study this further.

Another amusing fact in this picture is that we know that world-sheet instanton effects must vanish for the heterotic string on a K3 surface if "the spin connection if embedded in the gauge group". In other words, when our bundle becomes the tangent bundle. If this can be understood in the language of this paper then it should be quite a potent weapon in understanding the global structure of the moduli space further. It implies that there is a whole $\mathrm{O}(\Gamma_{4,20})\backslash\,\mathrm{O}(4,20)/(\mathrm{O}(4)\times\mathrm{O}(20))$ exact subspace of the moduli space coming from deformations of the K3 surface. This probes deeply into regions of the moduli space where the type IIA string becomes strongly coupled. Thus we may calculate instanton effects on the type IIA side too.

We should emphasize that there is a big difference between mirror symmetry and the map (4.1). The map given in (4.1) does not include the whole of $H^3(X,\mathbb{Z})$ and it does not contain the whole of $H^2(S_H,\mathbb{Z})$. In fact given the nature of the moduli space of strings on K3 surfaces [36] one might hope that such a map should include the total cohomology $H^*(S_H,\mathbb{Z})$. It would be very satisfying, as well as useful, if we could enlarge the identification of (4.1) to include these larger integral structures.

In conclusion, the moduli space of hypermultiplets of $N=2$ theories in four dimensions coming from compactifications of the type IIA string or the heterotic string contains some beautiful structures. These structures are similar to those of mirror symmetry but appear to be much more powerful. As well as providing insight into the nonperturbative properties of $N=2$ theories in four dimensions we may also learn some lessons about nonperturbative effects in general in string theory.

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A. Dimension formula from pryms

We wish to prove the following:[10]

Theorem 1 *Let W be the Weyl group of some given Lie group $\mathcal{G}$ and let Λ be the root lattice of $\mathcal{G}$. Let $\pi : \widetilde{C} \to B$ be a W-Galois Cameral cover, $\widetilde{C}$ be irreducible and B be a curve of genus 0. Then in the generic case*

$$\begin{aligned} l &= \tfrac{1}{2} \dim \mathrm{Hom}_W(\Lambda, H^1(\widetilde{C}, \mathbb{C})) \\ &= \tfrac{1}{2} b - \mathrm{rank}(\mathcal{G}) \,, \end{aligned} \tag{A.1}$$

where b is the number of branch points.

Consider the *representation ring* of W denoted $R(W)$. An element of this ring is a formal sum of the form

$$\sum n_\alpha [V_\alpha] \,, \tag{A.2}$$

where $n_\alpha \in \mathbb{Q}$ and V_α is an irreducible representation. There is an augmentation map

$$\varepsilon : R(W) \to \mathbb{Z} \,, \tag{A.3}$$

given by $\varepsilon([V_\alpha]) = \dim(V_\alpha)$.

Let us denote the simplicial chain complex of $\widetilde{C}$ by $\widetilde{S}_\bullet$ and the simplicial chain complex of B by $S_\bullet$. We may regard $\widetilde{S}_\bullet$ and $S_\bullet$ as $\mathbb{C}(W)$-modules (where the action of W on $S_\bullet$ is trivial).

The fibre of the covering π is a set of points corresponding to the Weyl chambers of W except over the points $p_j \in B$, $j = 1, \ldots, b$, where the covering is branched. At each p_j there is a monodromy action on the fibres given by an element of the Weyl group which we denote $\sigma_j \in W$.

It follows that, as $\mathbb{C}(W)$-modules,

$$\begin{aligned} \widetilde{S}_k &= S_k \otimes_{\mathbb{C}} \mathbb{C}(W) \qquad \text{for } k > 0 \\ \widetilde{S}_0 &= \left(S_0 - \bigcup_{i=1}^{b} \{p_j\} \right) \otimes_{\mathbb{C}} \mathbb{C}(W) \oplus \left(\bigoplus_{j=1}^{b} \{p_j\} \otimes_{\mathbb{C}} \mathbb{C}(W/\langle\sigma_j\rangle) \right) . \end{aligned} \tag{A.4}$$

One may show from standard methods that the Euler characteristic of the curve $\widetilde{C}$ is equal to the Euler characteristic of the chain complex $\widetilde{S}_\bullet$. Regarding $\widetilde{S}_\bullet$ as a chain complex of $\mathbb{C}(W)$-modules we may define $\hat{\chi} \in R(W)$ as a refinement of the usual Euler characteristic. The object $\hat{\chi}$ reduces to the usual Euler characteristic under the map ε. By the same argument as above $\hat{\chi}(\widetilde{C}) = \hat{\chi}(\widetilde{S}_\bullet)$. Thus

$$\hat{\chi}(\widetilde{C}) = \chi(B)[\mathbb{C}(W)] - b[\mathbb{C}(W)] + \sum_{j=1}^{b} [\mathbb{C}(W/\langle\sigma_j\rangle)] \,. \tag{A.5}$$

[10] This section was explained to me by R. Donagi and R. Hain.

It is now a straight-forward matter of using the representation theory of groups to determine l. Firstly one can show that $\dim \operatorname{Hom}_W(\Lambda, \mathbb{C}(W)) = \operatorname{rank}(\mathcal{G})$. The statement that our Cameral cover is generic amounts to the statement that the monodromy about any p_j corresponds to a simple Weyl reflection in a single root. Then $\dim \operatorname{Hom}_W(\Lambda, \mathbb{C}(W/\langle\sigma_j\rangle))$ is the dimension of the root space of $\mathcal{G}$ invariant under such a reflection which is $\operatorname{rank}(\mathcal{G}) - 1$.

We also know that $H^0(\widetilde{C})$ and $H^2(\widetilde{C})$ live in the identity representation of W. Thus $\dim \operatorname{Hom}_W(\Lambda, H^k(\widetilde{C})) = 0$ for $k = 0$ or $k = 2$. It follows that

$$\begin{aligned} \dim \operatorname{Hom}_W(\Lambda, H^1(\widetilde{C}, \mathbb{C})) &= -b(\operatorname{rank}(\mathcal{G}) - 1) - (2-b)\operatorname{rank}(\mathcal{G}) \\ &= b - 2\operatorname{rank}(\mathcal{G})\,. \qquad \square \end{aligned} \tag{A.6}$$

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